Engineering Notes

Modes and Frequencies of Pressurized Conical Shells

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THE purpose of the present paper is to describe briefly a technique, first proposed and used by the authors several years ago, for the direct numerical solution of a number of technically important problems of shells of revolution and circular plates. Using this technique, excellent results have been obtained, for example, for lateral vibration and buckling problems of circular plates such as rimmed turbine disks,[1, 2] and for static and dynamic problems of cylindrical, conical, and spherical shells.[3–8] Extension to other shapes, including the general shell of revolution and combinations of shells, such as spheres with coaxial nozzles, is a simple matter.[9, 10] The technique is applicable to both symmetric and unsymmetric problems of uniform and nonuniform plates and shells of revolution. The technique is also directly applicable to sectors of plates and shells, provided that the conditions along the radial or meridional boundaries permit representing the solution in the form of complete or truncated trigonometric series.

Anticipating the use of a digital computer, the governing equations are written as a system of first-order ordinary differential equations. For symmetric problems, the system will comprise six such equations. For unsymmetric problems using classical thin shell theory, the system will be of eighth order, and if the Kirchhoff assumptions are relaxed, the system of equations will be of tenth order. In contrast to the usual formulations of shell problems as prepared for computer solution, the dependent variables of the system of first order equations are the intrinsic or essential quantities and are, furthermore, precisely the displacements and tractions that are exposed at the boundaries. As a consequence, this technique has the advantage that all possible boundary conditions become simple and direct statements in terms of these dependent variables. A second and very significant advantage of this technique is that, in contrast to the usual formulations of nonuniform plate and shell problems, the equations involve only point values of the material properties and rigidities, and do not involve derivatives of these quantities.

To develop the system of equations appropriate to a particular configuration, one starts with the differential equations of motion or of equilibrium as given in any of the authoritative references,[11] and the formulas for the internal tractions in terms of the displacements. Expanding the loads and the dependent variables into series of trigonometric functions of the angle about the axis of revolution reduces the equations to ordinary differential equations. Introducing three new dependent variables, namely, the meridional or radial slope of the deformed surface, the effective membrane shear and the effective shear through the thickness (sometimes called the Kirchhoff shear), and operating only algebraically upon the resulting equations, yields the desired system of first order equations without derivatives of the mechanical quantities.

In the case of a conical shell,[10] the system of equations is as given below where the apex angle of the median surface is 2α. A point on the median surface is specified by the coordinates x and θ, x being measured from the apex along a generator, and the longitudinal angle θ being measured from an arbitrary meridional plane. The thickness of the shell is h and the coordinate z is measured normal to the median surface and is positive outward.

$$\left.\begin{aligned}
&dw^{(n)}/dx = S^{(n)}\\
&\frac{dS^{(n)}}{dx} = \frac{n\nu\cot\alpha}{x^2\sin\alpha}v^{(n)} + \frac{n^2}{x^2\sin\alpha}w^{(n)} - \frac{\nu}{x}S^{(n)} - \frac{1}{D}M_{xx}{}^{(n)}\\
&\frac{du^{(n)}}{dx} = -\frac{\nu}{x}u^{(n)} - \frac{n\nu}{x\sin\alpha}v^{(n)} - \frac{\nu\cot\alpha}{x}w^{(n)} + \frac{1}{E'}N_{xx}{}^{(n)}\\
&\frac{dv^{(n)}}{dx} = \frac{1}{G'[1 + (h^2\cot^2\alpha/12x^2)]}\times\\
&\quad\left\{Q_\theta{}^{(n)} + G'\left[1 + \frac{h^2\cot^2\alpha}{6x^2}\right]\frac{v^{(n)}}{x} + \frac{G'n}{x\sin\alpha}u^{(n)} - \right.\\
&\quad\left.\frac{D(1-\nu)\cot\alpha}{x^2\sin\alpha}n\,S^{(n)} + \frac{D(1-\nu)\cot\alpha}{x^2\sin\alpha}n\,w^{(n)}\right\}\\
&\frac{dN_{xx}{}^{(n)}}{dx} = -\frac{G'}{x}\left(2\frac{du^{(n)}}{dx} + \frac{n}{\sin\alpha}\frac{dv^{(n)}}{dx}\right) + \frac{3G'n}{x^2\sin\alpha}v^{(n)} +\\
&\left[\frac{G'}{x^2}\left(2 + \frac{n^2}{\sin^2\alpha}\right) - \rho h\,\omega^2\right]u^{(n)} + \frac{2G'\cot\alpha}{x^2}w^{(n)} - F_x{}^{(n)}\\
&\frac{dM_{xx}{}^{(n)}}{dx} = \frac{D(1-\nu)}{x}\left\{\frac{dS^{(n)}}{dx} - \frac{1}{x}\left(1 + \frac{2n^2}{\sin^2\alpha}\right)S^{(n)} - \right.\\
&\left.\frac{n\cot\alpha}{x\sin\alpha}\frac{dv^{(n)}}{dx} + \frac{3n\cot\alpha}{x^2\sin\alpha}v^{(n)} + \frac{3n^2}{x^2\sin^2\alpha}w^{(n)}\right\} - \frac{\rho h^3 w^2}{12}S^{(n)}\\
&\frac{dQz^{(n)}}{dx} = -\frac{Q_z{}^{(n)}}{x} - \frac{2n}{x^2\sin\alpha}M_{x\theta}{}^{(n)} + \frac{n^2}{x^2\sin^2\alpha}M_{\theta\theta}{}^{(n)} +\\
&\quad\frac{\cot\alpha}{x}N_{\theta\theta}{}^{(n)} - Fz^{(n)} - \rho h\omega^2 w^{(n)}\left(1 - \frac{n^2h^2}{12x^2\sin^2\alpha}\right) -\\
&\quad\bar{N}_x\frac{dS^{(n)}}{dx} - \bar{N}_\theta\left(\frac{S^{(n)}}{x} - \frac{n^2w^{(n)}}{x^2\sin^2\alpha} - \frac{n\cot\alpha}{x^2\sin\alpha}v^{(n)}\right)\\
&\frac{dQ_\theta{}^{(n)}}{dx} = -\frac{2Q_\theta{}^{(n)}}{x} + \frac{n\cot\alpha}{x^2\sin\alpha}M_{\theta\theta}{}^{(n)} - \frac{\cot\alpha}{x^2}M_{x\theta}{}^{(n)} +\\
&\quad\frac{n}{x\sin\alpha}N_{\theta\theta}{}^{(n)} - F_\theta{}^{(n)} - \rho h\omega^2 v^{(n)} + \frac{\rho h^3\omega^2 n^2\cot\alpha}{12x^2\sin\alpha}w^{(n)}
\end{aligned}\right\}\quad(1)$$

Presented at the AIAA Fifth Annual Structures and Materials Conference, Palm Springs, Calif., April 1–3, 1964 (no preprint number; published in bound volume of preprints of the meeting); revision received August 27, 1964.

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Also,

$$\left.\begin{aligned}
N_{\theta\theta}{}^{(n)} &= E'\left\{\frac{n}{x\sin\alpha}\,v^{(n)} + \frac{u^{(n)} + w^{(n)}\cot\alpha}{x} + \nu\,\frac{du^{(n)}}{dx}\right\} \\
N_{x\theta}{}^{(n)} &= G'\left\{\frac{dv^{(n)}}{dx} - \frac{v^{(n)}}{x} - \frac{n}{x\sin\alpha}\,u^{(n)}\right\} \\
M_{\theta\theta}{}^{(n)} &= -D\left\{-\frac{n^2}{x^2\sin^2\alpha}\,w^{(n)} + \frac{1}{x}\,S^{(n)} - \frac{n\cot\alpha}{x^2\sin\alpha}\,v^{(n)} + \nu\,\frac{dS^{(n)}}{dx}\right\} \\
M_{x\theta}{}^{(n)} &= D(1-\nu)\left\{\frac{n}{x\sin\alpha}\,S^{(n)} - \frac{n}{x^2\sin\alpha}\,w^{(n)} + \frac{\cot\alpha}{x}\left[\frac{1}{2}\,\frac{dv^{(n)}}{dx} - \frac{v^{(n)}}{x}\right]\right\}
\end{aligned}\right\} \quad (2)$$

In the preceding equations, ω is the circular frequency of vibration, F_x, F_θ, and F_z are the components of surface tractions, and barred quantities are initial membrane stresses; also,

$$\begin{aligned}
Q_z{}^{(n)} &= N_{xz}{}^{(n)} + (n/x\sin\alpha)\,M_{x\theta}{}^{(n)} \\
Q_\theta{}^{(n)} &= N_{x\theta}{}^{(n)} + (\cot\alpha/x)M_{x\theta}{}^{(n)}
\end{aligned} \quad (3)$$

$$\begin{gathered}
E' = Eh/(1-\nu^2) \qquad D = Eh^3/12(1-\nu^2) \\
G' = Eh/2(1+\nu)
\end{gathered} \quad (4)$$

Numerical Integration

As has been previously shown, the differential equations governing the behavior of the shell have been reduced to a set of eight first order differential equations. The form of these equations suggests that they may be directly amenable to a numerical integration using a scheme such as the Runge-Kutta fourth order process. In principle this is true; however, since numerical integration techniques are basically initial value techniques, some modification is required. This modification is described below.

Because of the linearized nature of the governing differential equations, the principle of superposition is applicable. This forms the key to the technique. Assume, for the present, that the natural frequency is known. The problem can then be thought of as an initial value problem with four of the intrinsic variables specified by the initial boundary conditions (e.g., the four components of displacement). The initial values of the four remaining intrinsic variables must be selected so that the four final boundary conditions are satisfied by the solution. We may, for purposes of the present discussion, denote the four specified initial values by x_5, x_6, x_7, x_8 and the remaining four dependent variables by x_1, x_2, x_3, x_4.

We now construct, by numerical integration, a set of five solutions that we number 0, 1, 2, 3, 4. For the first of these (solution 0) we use the specified initial values, x_5, x_6, x_7, x_8 and we take each of the four remaining values to be zero. For this solution we retain the nonhomogeneous terms (loads, temperature, etc.) in the differential equations. For the next four solutions, we delete the nonhomogeneous terms and take x_5, x_6, x_7, x_8 to be zero. The other four initial conditions are selected in a linearly independent fashion.

The values of the eight dependent variables at the end of the integration are denoted by $y_j{}^i$ where the superscript refers to the solution number ($i = 0, 1, 2, 3, 4$) and the subscript indicates the intrinsic variable ($j = 1, 2, \ldots, 8$).

The correct solution is then a linear combination of the five solutions obtained. If we let f_j be the four known final boundary conditions, and let a^i be the correct values of the unknown initial conditions x_1, x_2, x_3, x_4; then the initial and final conditions will be satisfied if the a^i's satisfy the following set of linear algebraic equations:

$$y_j{}^0 + \sum_{i=1}^{4} a^i y_j{}^i = f_j$$

The problem of determining the natural frequency can now be discussed. It is apparent from the previous discussion, that if the natural frequency is known, or guessed with sufficient accuracy, the boundary conditions can be satisfied, otherwise it will be impossible to satisfy all of the boundary conditions. It is then apparent that we have five unknown quantities for starting the integration. The obvious solution is to guess one of the five unknowns (e.g., ω) and see if the boundary conditions can all be satisfied with sufficient accuracy. This obvious method suffers from an equally obvious difficulty, specifically, the difficulty in guessing how much and in what direction to modify the guess to improve the satisfaction of the boundary conditions. Fortunately, in many problems of practical importance it is possible to circumvent this difficulty by a simple expedient.

In the cases where we are considering free vibrations, i.e., no nonhomogeneous load terms, the following technique has proved valuable. The five solutions for the four variables specified by the final boundary conditions form a set of four linear algebraic equations. Under the stated conditions we note that the final boundary conditions are specified by a set of linear algebraic equations:

$$\sum_{i=1}^{4} a^i y_j{}^i = 0$$

A necessary condition that this set of equations have a nontrivial solution is

$$\mathrm{Det}\, y_j{}^i = 0$$

Thus, the value of the determinant of $y_j{}^i$ gives a measure of the degree of satisfaction of the boundary conditions.

Once the determinant has a sufficiently small value it is possible to determine the ratios of the unknown initial condition. The mode shape is determined by making a final integration using the four a priori known initial conditions and arbitrarily assuming one of the unknown initial conditions; the other three unknown initial conditions are calculated from the ratios required to cause vanishing of the determinant. Of course, the solutions obtained in this manner do not necessarily represent the actual solution of the vibrating cone but the actual solution multiplied by a scale factor.

If the problem is not to determine the natural frequencies and mode shapes, but to determine the stresses and displacements of a cone under given loads and boundary conditions, the final integration is carried out by using the known initial conditions, the computed initial conditions and the differential equations with the nonhomogeneous terms included.

One final difficulty remains in solving this type of problem. In thin shells, it is well known that the solutions can often be expressed as a superposition of a "membrane" solution and a "boundary layer" or "edge effect" solution. Physically, the effects of edge disturbances are rapidly attenuated. In the case of a cylinder considered as a limiting case of a cone, the edge disturbances are attenuated exponentially. Roughly, the same type of attenuation exists in cones and also in the general theory of thin shells. In principle, the technique previously described will lead to correct results; in practice, however, since the integrations are limited to a finite number of significant figures (8 or 16), the truncation errors rapidly obscure the edge effects. In other words, we are trying to determine initial conditions from quantities which are, as far as the computer is concerned, zero.

To overcome these difficulties, a technique has been developed by the authors and applied to various shell problems. Without the necessary modification, it is found that the solu-

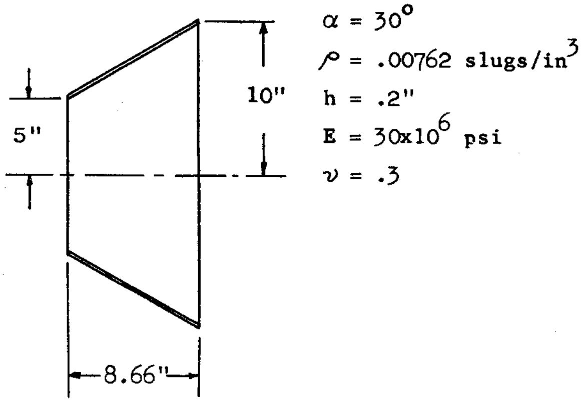

Fig 1. Shell geometry for example.

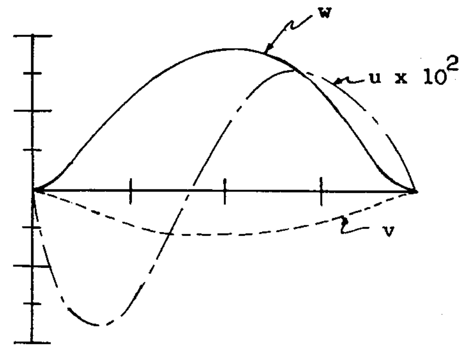

Fig 2. Mode shapes at ω = 718.4 cps for n = 2.

tions will start correctly and proceed properly for a while; but, nearing the far end, the solutions begin to wander from the correct values. When appropriate modifications are utilized, the solutions are found to be accurate over the entire range of integration.

It has been found necessary to break the shell into a number of segments and perform the type of integration described previously for each segment. From these integrations, influence coefficients are formed, showing the effect of unit displacements of the previous segment. These influence coefficients are then used in the final integration.

For the present paper a computer program has been written in Fortran II for the IBM 7090 digital computer with 32 k, storage capacity. The program is completely automatic in the sense that, for given input data for the shell geometry and boundary conditions, and an initial guess at the natural frequency (actually, an initial condition guess could easily be incorporated in the program), the value of the determinant of final conditions is evaluated and a new estimate made. The iterative process continues until the value of the determinant approaches zero to satisfactory accuracy. The natural frequencies and mode shapes are then automatically given as output data.

The running time is about one minute per iteration with five iterations usually sufficient for determination of one frequency and associated mode shape.

Illustrative Example

As an illustrative example, one of the unsymmetrical resonant frequencies and the associated mode have been calculated for the shell segment shown in Fig. 1. It is assumed that the complete shell is pressurized to 100 psi, and both ends of the segment are then rigidly clamped. The lowest calculated frequency in the third circumferential mode (n = 2; four meridional nodal lines) was found to be 718.4 cps. The associated calculated mode shape is shown in Fig. 2.

The clamped boundary conditions were chosen only for illustrative purposes. The theory and method permit the boundary conditions to be specified in a wholly arbitrary manner, in terms of any consistent set of edge tractions and edge displacements. Thus, free and guided edges, elastically supported edges, and mass-carrying edges may be handled readily. The method is not restricted to the determination of the lowest frequency associated with any specified number of meridional nodal lines, but may also be used to determine the higher frequencies and corresponding modes including the symmetrical cases.

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